Chapter 1

What This Book Is About and How to Read It

1.1 "Exercises" vs. "Problems"

This is a book about mathematical problem solving. We make three assumptions about you, our reader:

- You enjoy math.
- You know high-school math pretty well, and have at least begun the study of "higher mathematics" such as calculus and linear algebra.
- You want to become better at solving math problems.

First, what is a **problem**? We distinguish between **problems** and **exercises**. An exercise is a question that you know how to resolve immediately. Whether you get it right or not depends on how expertly you apply specific techniques, but you don't need to puzzle out what techniques to use. In contrast, a problem demands much thought and resourcefulness before the right approach is found. For example, here is an exercise.

Example 1.1.1 Compute 5436^3 without a calculator.

You have no doubt about how to proceed—just multiply, carefully. The next question is more subtle.

Example 1.1.2 Write

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{99\cdot 100}$$

as a fraction in lowest terms.

At first glance, it is another tedious exercise, for you can just carefully add up all 99 terms, and hope that you get the right answer. But a little investigation yields something intriguing. Adding the first few terms and simplifying, we discover that

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}=\frac{2}{3},$$

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}=\frac{3}{4},$$
$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}=\frac{4}{5},$$

which leads to the **conjecture** that for all positive integers n,

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}.$$

So now we are confronted with a **problem**: is this conjecture true, and if so, how do we *prove* that it is true? If we are experienced in such matters, this is still a mere exercise, in the technique of mathematical induction (see page 45). But if we are not experienced, it is a problem, not an exercise. To solve it, we need to spend some time, trying out different approaches. The harder the problem, the more time we need. Often the first approach fails. Sometimes the first dozen approaches fail!

Here is another question, the famous "Census-Taker Problem." A few people might think of this as an exercise, but for most, it is a problem.

Example 1.1.3 A census-taker knocks on a door, and asks the woman inside how many children she has and how old they are.

"I have three daughters, their ages are whole numbers, and the product of the ages is 36," says the mother.

"That's not enough information," responds the census-taker.

"I'd tell you the sum of their ages, but you'd still be stumped."

"I wish you'd tell me something more."

"Okay, my oldest daughter Annie likes dogs."

What are the ages of the three daughters?

After the first reading, it seems impossible—there isn't enough information to determine the ages. That's why it is a problem, and a fun one, at that. (The answer is at the end of this chapter, on page 12, if you get stumped.)

If the Census-Taker Problem is too easy, try this next one (see page 75 for solution):

Example 1.1.4 I invite 10 couples to a party at my house. I ask everyone present, including my wife, how many people they shook hands with. It turns out that everyone questioned—I didn't question myself, of course—shook hands with a different number of people. If we assume that no one shook hands with his or her partner, how many people did my wife shake hands with? (I did not ask myself any questions.)

A good problem is mysterious and interesting. It is mysterious, because at first you don't know how to solve it. If it is not interesting, you won't think about it much. If it is interesting, though, you will want to put a lot of time and effort into understanding it.

This book will help you to investigate and solve problems. If you are an inexperienced problem solver, you may often give up quickly. This happens for several reasons.

- You may just not know how to begin.

- You may make some initial progress, but then cannot proceed further.
- You try a few things, nothing works, so you give up.

An experienced problem solver, in contrast, is rarely at a loss for how to begin investigating a problem. He or she[1] confidently tries a number of approaches to get started. This may not solve the problem, but some progress is made. Then more specific techniques come into play. Eventually, at least some of the time, the problem is resolved. The experienced problem solver operates on three different levels:

Strategy: Mathematical and psychological ideas for starting and pursuing problems.

Tactics: Diverse mathematical methods that work in many different settings.

Tools: Narrowly focused techniques and "tricks" for specific situations.

(spoilers)

Solution to the Census-Taker Problem

The product of the ages is 36, so there are only a few possible triples of ages. Here is a table of all the possibilities, with the sums of the ages below each triple.

(1,1,36)	(1,2,18)	(1,3,12)	(1,4,9)	(1,6,6)	(2,2,9)	(2,3,6)	(3,3,4)
38	21	16	14	13	13	11	10

Aha! Now we see what is going on. The mother's second statement ("I'd tell you the sum of their ages, but you'd still be stumped") gives us valuable information. It tells us that the ages are either $(1,6,6)$ or $(2,2,9)$, for in all other cases, knowledge of the sum would tell us unambiguously what the ages are! The final clue now makes sense; it tells us that there *is* an oldest daughter, eliminating the triple $(1,6,6)$. The daughters are thus 2, 2 and 9 years old. ■

[8] Named after Paul Halmos, a mathematician and writer who popularized its use.